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Chern-Simons gravity and holographic anomalies*

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Abstract: We present a holographic treatment of Chern-Simons (CS) gravity theories in odd dimensions. We construct the associated holographic stress tensor and calculate the Weyl anomalies of the dual CFT.

Keywords: AdS-CFT and dS-CFT Correspondence, Anomalies in Field and String Theories.

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1. Introduction and summary

The existence of a duality between gravitational theories in odd dimensions and conformal field theories living on the boundary was first indicated by the remarkable observation of Brown and Henneaux [1] that the asymptotic group of symmetries of 2+1 gravity with a negative cosmological constant $\Lambda = -2/l^2$ is the two-dimensional conformal group with a non-vanishing central charge

$$c = \frac{3l}{2G}\,.$$

Once it was understood that three-dimensional gravity can be written as a Chern-Simons(CS) theory [2, 3] and that generically three dimensional CS theories are related to two dimensional conformal field theories [4], a more explicit relation underlying the Brown-Henneaux argument became available [5] (see [6] for a previous attempt).

With the arrival of Maldacena's conjecture [7], these results became special cases of a much larger connection between gravitational and field theories.

The three dimensional CS gravity theory has some very special features not having propagating degrees of freedom. Its generalization to higher (odd) dimensions [8] is a fully interacting theory which makes its study much more difficult. The corresponding supergravity theories were formulated in [8, 9] and the hamiltonian structure of the theory was studied in [10].

In the present work we study a few basic issues related to a holographic interpretation of higher dimensional CS theories. We construct the holographic stress tensor and calculate the conformal anomalies, as first done in [11] for standard gravity, of the (even dimensional) conformal field theory dual to the CS theory. The calculations have rather unusual features due to two characteristic properties of CS theories:

(a) Even though the CS action when rewritten in metric form contains higher powers of the curvatures, the equations of motion are polynomial in the curvatures, containing at most second derivatives of the metric.

(b) The AdS solution is n-th degenerate in $d = 2n + 1$ dimensions and therefore the expansion around the solution starts with a $n + 1$-th order term.

Due to property (b) the standard Fefferman-Graham expansion [12] breaks down and the usual methods of evaluating the conformal anomalies cannot be used. We calculate the anomalies in three different ways:

(i) We use the coefficients of the CS action in integer dimension in dimensional regularization when property b) is not obeyed and then we take the limit to integer dimension. The values of the anomalies are obtained from the general formulae for actions with higher powers of the curvatures [13, 14, 15].

(ii) We use a dimensionally continued CS action where the dimension dependent coefficients are tuned in order that a), b) are obeyed and we calculate the anomalies from the equations of motion.

(iii) In integer dimensions, we derive the stress tensor from the equations of motion, and find a general formula valid for all n. We also calculate the stress tensor in the hamiltonian formalism.

For (i) and (ii) we only present results for $n = 2$, but for (iii) we give explicit results for any n.

All the different ways of calculating the anomalies agree and give the results:

(a) All type B anomalies vanish (we remind that their number increases with the dimension)

(b) The type A anomalies (one in each even dimension) are nonzero and consistent with the universal formula of [16] for the specific CS action.

Feature (a) restricts a direct holographic interpretation of the CS action. Through the diffeo Ward identities the type B anomalies are related to lower order correlators of energy momentum tensors and one of them to the two point function. Its vanishing cannot happen in a unitary conformal theory. On the other hand (b) shows that the CS gravitational actions provide the correct analogue of the CS gauge actions which generate the chiral anomalies for generating the type A trace anomalies.

The outline of the paper is as follows. In the next section we review the features of CS gravity which are relevant for the further developments. In section 3 we present the holographic analysis in the spirit of refs. [16, 17, 15]. For this we need to define a

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dimensionally continued CS gravity action. We do this explicitly for the case appropriate to a four-dimensional boundary theory. In integer dimensions the powerful method of differential forms, in which the CS theory is most naturally formulated, leads to a very efficient method, based on the equations of motion, to derive the holographic stress tensor for Chern-Simons gravity in any odd dimension. This is done in section 4. In section 5 we present a hamiltonian derivation of the stress tensor which is applicable for any gravity theory. As a first demonstration of this method we reproduce the known result for ordinary gravity (cf. also [18, 19]). Next we apply it to CS gravity which turns our to be much simpler. As a final application of our formula for the stress tensor we use it to compute the mass of CS black holes.

2. Review of Chern-Simons gravity

The action of euclidean CS gravity in $2n+1$ dimensions is [8]

$$I_{2n+1} = \int_{M_{2n+1}} \omega_{2n+1} \,. \tag{2.1}$$

ω_{2n+1} is a CS $(2n+1)$-form for the group $\mathrm{SO}(1,2n+1)$, i.e.

$$d\omega_{2n+1} = F^{A_1 A_2} \wedge \cdots \wedge F^{A_{2n+1} A_{2n+2}} \epsilon_{A_1 \ldots A_{2n+2}} \tag{2.2}$$

where F^{AB} is the curvature two-form. This action is invariant under gauge transformations, up to a boundary term.

To exhibit the gravitational character of the CS action one splits the $\mathrm{SO}(1,2n+1)$ indices $A=(0,a)$ and decomposes the gauge potential according to[1]

$$A = \frac{1}{2} A^{AB} J_{AB} = \frac{1}{2} \hat{\omega}^{ab} J_{ab} + \hat{e}^a P_a \tag{2.3}$$

where

$$A^{0a} = \hat{e}^a \,, \qquad J_{0a} = P_a \,, \tag{2.4}$$

$$F = \frac{1}{2} F^{AB} J_{AB} = \frac{1}{2} (\hat{R}^{ab} + \hat{e}^a \hat{e}^b) J_{ab} + \hat{T}^a P_a \tag{2.5}$$

and

$$\begin{aligned} \hat{R}^{ab} &= d\hat{\omega}^{ab} + \hat{\omega}^a{}_c \hat{\omega}^{cb} \,, \\ \hat{T}^a &= d\hat{e}^a + \hat{\omega}^a{}_b \hat{e}^b \,. \end{aligned} \tag{2.6}$$

Using this decomposition the action becomes, up to a boundary term,

$$\begin{aligned} I_{2n+1} &= \int_{M_{2n+1}} \epsilon_{\alpha_1 \ldots \alpha_{2n+1}} \sum_{p=0}^{n} \frac{1}{2(n-p)+1} \binom{n}{p} \hat{R}^{\alpha_1 \alpha_2} \wedge \cdots \wedge \hat{R}^{\alpha_{2p-1} \alpha_{2p}} \wedge \hat{e}^{\alpha_{2p+1}} \wedge \ldots \wedge \hat{e}^{\alpha_{2n+1}} \\ &= \int_{M_{2n+1}} d^{2n+1}x \sqrt{\hat{g}} \sum_{p=0}^{n} \binom{n}{p} [2(n-p)]! \, \hat{E}_{2p} \,. \end{aligned} \tag{2.7}$$

[1] Hatted quantities are defined in the $(2n+1)$-dimensional bulk. Unhatted symbols will be used to below and refer to quantities defined on the $2n$-dimensional boundary.

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In (2.7) the expressions $\hat{E}_{2n}$ are the Euler densities

$$E_{2n} \equiv \tfrac{1}{2^n} R_{i_1 j_1 k_1 l_1} \cdots R_{i_n j_n k_n l_n} \epsilon^{i_1 j_1 i_2 j_2 \cdots i_n j_n} \epsilon^{k_1 l_1 k_2 l_2 \ldots k_n l_n} = R^n + \cdots \tag{2.8}$$

The integral of $\int_M \sqrt{g}\, E_{2n}$ over a $(2n)$-dimensional manifold without boundary is a topological invariant of M. While the expression in the first line is only meaningful in $2n$ dimensions, the expression in the second line can be defined in any dimension. For instance, for $n = 2$ one finds explicitly

$$E_4 = \frac{1}{4} R_{ijkl} R_{mnpq} \epsilon^{ijmn} \epsilon^{klpq} = R^2 - 4R^{ij} R_{ij} + R_{ijkl} R^{ijkl} \,. \tag{2.9}$$

For many purposes the form of the action as written in the first line of (2.7) is more convenient, in particular for formal manipulations. However it is the second form which allows continuation to non-integer dimensions.

If one allows in (2.7) for arbitrary relative coefficients for the $\hat{E}_{2p}$ one arrives at Lovelock gravity. It is, however, only for the special coefficients, namely those of CS-gravity, that the manifest $\mathrm{SO}(2n+1)$ symmetry is enhanced to $\mathrm{SO}(1, 2n+1)$. This makes CS gravity in many respects very 'non-generic', as we will see in the following.

From (2.7) one derives the following equation of motion for the vielbein $\hat{e}^a_\mu$:

$$\epsilon_{\mu_1 \ldots \mu_{2n+1}} F^{\mu_1 \mu_2} \wedge \cdots \wedge F^{\mu_{2n-1} \mu_{2n}} = 0 \tag{2.10}$$

where

$$F^{\mu\nu} = \hat{R}^{\mu\nu} + dx^\mu \wedge dx^\nu \,. \tag{2.11}$$

The equation of motion for the spin connection ω is solved imposing the torsion constraint $\hat{T} = 0$. This is the general solution for $n = 1$. In more than three dimensions there are other solutions. We will, however, always impose $\hat{T} = 0$ and thus the only degrees of freedom are those of the metric.

As it is clear from (2.11) the vanishing of $F^{\mu\nu}$ is equivalent to the $2n + 1$ dimensional metric being AdS. Then (2.10) shows that after the torsion constraint is taken into account AdS is an n fold degenerate solution of the equations of motion. As a consequence an expansion around the AdS solution will start with $n + 1$ order terms.

3. Dimensionally continued Chern-Simons gravity

In this section we will restrict ourselves to an action with at most four-derivatives. This is appropriate for the dimensional continuation of five-dimensional CS gravity. The most general such action is

$$S = \int d^{d+1}x \sqrt{G} \left(\hat{R} - 2\Lambda + \alpha \hat{R}^2 + \beta \hat{R}_{\mu\nu} \hat{R}^{\mu\nu} + \gamma \hat{R}_{\mu\nu\rho\sigma} \hat{R}^{\mu\nu\rho\sigma} \right) . \tag{3.1}$$

One needs[2] $\Lambda = -\frac{1}{2} d(d-1) + \frac{1}{2}(d-3) \left(\alpha d^2 (d+1) + \beta d^2 + 2\gamma d \right) < 0$ for AdS_{d+1} to be a solution of the equations of motion. This is required for the AdS/CFT correspondence.

[2] We use the following sign convention for the Riemann tensor: $[\nabla_\mu, \nabla_\nu] V_\rho = R_{\mu\nu\rho}{}^\sigma V_\sigma$.

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The Weyl anomaly for the dual conformal field theory in $d = 4$ was computed in [13, 14, 15] with the result

$$\langle T^i_i \rangle = \frac{1}{8}\left\{(1 - 40\alpha - 8\beta + 4\gamma)C^2 - (1 - 40\alpha - 8\beta - 4\gamma)E_4\right\} . \tag{3.2}$$

For $d = 4$, $\beta = -4\alpha = -4\gamma = -1$ and $\Lambda = -3$ (3.1) becomes the action of CS gravity in five dimensions. For these values of the parameters the anomaly (3.2) is purely type A. However, it is à priori not obvious that this result is reliable since it was obtained under the condition that the generic FG-expansion is valid.

We have mentioned in the introduction the two special features of this action, namely that the equations of motion contain no higher than second derivatives of the metric and that when expanded around AdS_5 the expansion starts at cubic order. The first feature is maintained as long as $\beta = -4\alpha = -4\gamma$. This means that at each order in curvatures they appear in the Euler combination. The second feature requires

$$\Lambda = -\frac{1}{4}d(d-1)\,, \qquad \alpha = \gamma = -\frac{1}{4}\beta = \frac{1}{2(d-2)(d-3)} \tag{3.3}$$

as a tedious calculation reveals. The action (3.1) with the choice (3.3), which is the dimensionally continued five-dimensional CS action, will be the starting point of our analysis.

We make the FG expansion for the bulk metric $G_{\mu\nu}$, i.e. we make the Ansatz [12]

$$ds^2 = \frac{1}{4}\left(\frac{dr}{r}\right)^2 + \frac{1}{r}g_{ij}(x,r)dx^i dx^j \tag{3.4}$$

with

$$g_{ij}(x,r) = \sum_{n=0}^{\infty} \overset{(n)}{g}_{ij}(x)r^n \,. \tag{3.5}$$

The coefficients $g^{(n)}{}_{ij}$ are to a large extent fixed by the symmetries, i.e. invariance of (3.4) under so-called PBH transformations [16]. To fix them completely one inserts the Ansatz into the equations of motion and solves for the $g^{(n)}{}_{ij}$ recursively. They can be expressed in terms of curvature tensors constructed from $g^{(0)}{}_{ij}$ with a total of $2n$ derivatives of the metric. For generic gravitational actions, e.g. (3.1), the $g^{(n)}{}_{ij}$ are all local as long as one stays away from integer dimensions. For instance, the PBH transformations completely fix the local part of $g^{(1)}{}_{ij}$ to

$$\overset{(1)}{g}{}^{\rm loc}_{ij} = -\frac{1}{(d-2)}\left(\overset{(0)}{R}_{ij} - \frac{1}{2(d-1)}\overset{(0)}{R}\,\overset{(0)}{g}_{ij}\right). \tag{3.6}$$

The only freedom left is an additive term which is a tensor built from $g^{(0)}{}_{ij}$ which is invariant under Weyl transformations of $g^{(0)}{}_{ij}$ and transforms homogeneously of order -2 under a constant rescaling of the coordinates. Clearly this cannot be a finite polynomial in the curvature tensors. Terms containing tensors constructed from C_{ijkl}, the Weyl tensor, multiplied with powers of $\sqrt{C^2}$ have the right transformation properties. We will see that such a term is required in CS gravity.

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Before writing down the equations of motion we want to compute the Weyl anomaly for CS-gravity for $d = 4$. From [16] we know that it is given (in $d = 2n$) by the $\mathcal{O}(r^{-1})$ coefficient of the expansion of the gravitational action. For CS gravity and $n = 2$ one finds

$$\langle T_i^i \rangle_{CS} = \frac{1}{4} E_4 \,, \tag{3.7}$$

which agrees with what we found before. While the previous derivation used the explicit form for $g^{(1)}{}_{ij}$ this derivation does not need the explicit form of any of the coefficients $g^{(n)}$. This is a consequence of the choice (3.3).

We now return to the equations of motion. For (3.1) with the choice (3.3) one finds that at lowest orders in r the equations of motion are identically satisfied. The first non-trivial equations are at $\mathcal{O}(r^0)$ for the (rr) components and at $\mathcal{O}(r)$ for the (ij) and (ir) components. At these orders the $g^{(2)}$ dependence drops out identically as a consequence of the choice (3.3).

The equations can now be written as the definition of the non-local conserved energy-momentum tensor T_{ij}, its trace and conservation. One finds

$$\begin{aligned} T_{ij} &= \frac{1}{24(d-4)} \left(C_{iklm} C_j{}^{klm} - \frac{1}{4} C^2 g_{ij} \right) + \text{finite as } d \to 4 \,, \\ \nabla^i T_{ij} &= 0 \,, \\ g^{ij} T_{ij} &= -\frac{1}{4} (R^2 - 4 R_{ij} R^{ij} + R_{ijkl} R^{ijkl}) \end{aligned} \tag{3.8}$$

where T_{ij} is the following expression in terms of $g^{(1)}{}_{ij}$:

$$\begin{aligned} T_{ij} = \frac{1}{6} \Big\{ & (d-3) \Big(2 \overset{(1)}{g}{}^2_{ij} - 2 \,\mathrm{tr}(\overset{(1)}{g}) \overset{(1)}{g}_{ij} + (\mathrm{tr}\, \overset{(1)}{g})^2 g_{ij} - \mathrm{tr}(\overset{(1)}{g}{}^2) g_{ij} \Big) + \\ & + 2 \left(R_i{}^k \overset{(1)}{g}_{jk} + R_j{}^k \overset{(1)}{g}_{ik} \right) + 2 R_{ikjl} \overset{(1)}{g}{}^{kl} + R \,\mathrm{tr}(\overset{(1)}{g}) g_{ij} - \\ & - 2 R_{kl} \overset{(1)}{g}{}^{kl} g_{ij} - R \overset{(1)}{g}_{ij} - 2 \,\mathrm{tr}(\overset{(1)}{g}) R_{ij} \Big\} \,. \end{aligned} \tag{3.9}$$

In these expressions all curvature tensors are computed with $g^{(0)}$ which is also used to raise indices, g_{ij} stands for $g^{(0)}{}_{ij}$ and C is they Weyl tensor which is totally traceless in d-dimensions.[3] The first term on the r.h.s. of eq. (3.8) becomes 0/0 in $d = 4$ due to a special identity.

Eq. $(3.8)_1$ together with (3.9) determine $g^{(1)}$. We have not obtained a closed expression for $g^{(1)}$ itself but clearly it cannot be local in $d = 4$ (see also the discussion in [15]). The unique local expression for $g^{(1)}{}_{ij}$ which solves the PBH equation has to be augmented by a non-local piece which is invariant under Weyl transformations. We write

$$\overset{(1)}{g}_{ij} = \overset{(1)}{g}{}^{\mathrm{loc}}_{ij} + \Delta_{ij} \,. \tag{3.10}$$

[3] On the r.h.s. of the first of eq. (3.8) the finite piece, which is traceless in $d = 4$, is absent if one instead interprets C as the Weyl-tensor which is traceless in $d = 4$ but with the range of its indices extended to d.

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Using this definition, the (ij) component of the equation motion at $\mathcal{O}(r)$ can be written in the form

$$C^{(d)}_{ikjl}\Delta^{kl}+\frac{1}{2}(d-3)\left\{2\Delta^2_{ij}-2(\mathrm{tr}\,\Delta)\Delta_{ij}+(\mathrm{tr}\,\Delta)^2\overset{(0)}{g}_{ij}-\mathrm{tr}(\Delta^2)\overset{(0)}{g}_{ij}\right\}=$$
$$=\frac{1}{8(d-4)}\left\{C^{(d)2}_{ij}-\frac{1}{4}C^{(d)2}\overset{(0)}{g}_{ij}\right\}. \qquad (3.11)$$

What we found here is reminiscent of the situation for the generic gravitational action: there the equations of motion determine $g^{(2)}{}_{ij}$ which is local but has a pole at $d=4$. However, $g^{(2)}{}_{ij}$ consists of two cohomologically non-trivial pieces: one term which is the same (up to an overall coefficient) as the r.h.s. of (3.11) and an other which has a genuine pole (i.e. finite residue) at $d=4$. They are related to type A and type B anomalies. Here the cohomologically non-trivial information resides in the expression on the r.h.s. of (3.11). The significance of the particular combination of non-local terms Δ_{ij} is not clear to us other that it produces a local expression.

We remark that in a holographic interpretation the Conformal Field Theory living on the boundary will be necessarily non-unitary. This is a consequence of the vanishing of the type B anomaly. As it is well known this anomaly can be related to the correlator of two energy momentum tensors which cannot vanish in a unitary theory.

4. Chern-Simons stress tensor. Integer dimensions analysis

In this and the following section we will rederive the results of section 3 using the equations of motion and action in integer dimensions. We use in this section the following form of the Chern-Simons equations of motion (c.f. (2.10))

$$\varepsilon_{\mu\nu\lambda\rho\sigma}(\hat{R}^{\mu\nu}+dx^{\mu}\wedge dx^{\nu})\wedge(\hat{R}^{\lambda\rho}+dx^{\lambda}\wedge dx^{\rho})=0 \qquad (4.1)$$

where

$$\hat{R}^{\mu\nu}=\frac{1}{2}\hat{R}^{\mu\nu}{}_{\lambda\rho}dx^{\lambda}\wedge dx^{\rho} \qquad (4.2)$$

is the 2-form Riemann tensor. We shall see that this notation provides a powerful way to identify the stress-tensor in Chern-Simons gravity.

This section is organized as follows. We first review some standard results and continue by making the connection with the FG expansion (3.4) and (3.5). We then treat the five-dimensional case and recover the results of the previous section. We then apply the formalism to three and seven dimensional CS gravity, and finally provide a general formula for the Chern-Simons holographic stress tensor valid in an arbitrary dimension $D=2n+1$.

4.1 The FG expansion and Gauss-Codazzi equations

Consider the space-time metric in normal coordinates

$$ds^2=N^2(r)dr^2+h_{ij}(r,x^i)dx^idx^j \qquad (4.3)$$

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and introduce the standard notation

$$K_{ij} = -\frac{1}{2N} h'_{ij} \tag{4.4}$$

where the prime denotes derivatives w.r.t. r. The space-time curvature can be decomposed in the Gauss-Codazzi form

$$\begin{aligned}
\hat{R}^{ir}{}_{kl} &= \frac{1}{N}(K^i{}_{k/l} - K^i{}_{l/k}) \,, \\
\hat{R}^{ir}{}_{jr} &= \frac{1}{N} K^{i\prime}_{j} - K^i{}_l K^l{}_j \,, \\
\hat{R}^{ij}{}_{kl} &= R^{ij}{}_{kl} - K^i{}_k K^j{}_l + K^i{}_l K^j{}_k
\end{aligned} \tag{4.5}$$

where / represents the $2n$-dimensional covariant derivative in the metric h_{ij}. Introducing the curvature 2-form R^{ij} and extrinsic curvature 1-form $K^i = K^i{}_j dx^j$, these expressions can be rewritten more compactly as

$$\begin{aligned}
\hat{R}^{ir} &= -\frac{1}{N} DK^i + \left(\frac{1}{N} K^{i\prime} - K^i{}_j K^j\right) \wedge dr \,, \\
\hat{R}^{ij} &= R^{ij} - K^i \wedge K^j + N(K^{i/j} - K^{j/i}) \wedge dr \,.
\end{aligned} \tag{4.6}$$

We now make contact with (3.4) by making a definite choice of the radial coordinate, i.e.,

$$N = \frac{1}{2r} \tag{4.7}$$

and introduce the metric g_{ij} as

$$h_{ij}(r, x^i) = \frac{1}{r} g_{ij}(r, x^i) \,. \tag{4.8}$$

Then, it follows,

$$K_{ij} = \frac{1}{r} g_{ij} - g'_{ij} \ \Rightarrow \ K^i = dx^i - r k^i \tag{4.9}$$

where we have defined $k_{ij} = g'_{ij}$ and $k^i{}_j = g^{ik} g'_{kj}$.

Since the Christoffel symbols are invariant under constant rescaling of the metric, and multiplying by r is a constant rescaling, the covariant derivatives are not altered by the field redefinition $h_{ij} \to g_{ij}$.

Recall now the definition of the SO$(1,5)$ curvature which enters in the CS equations of motion (4.1)[4]

$$F^{\mu\nu} = \hat{R}^{\mu\nu} + dx^\mu \wedge dx^\nu \,. \tag{4.10}$$

By direct computation we find

$$\begin{aligned}
F^{ir} &= r^2 [2Dk^i - (2k^{i\prime} + k^i{}_j k^j) \wedge dr] \,, \\
F^{ij} &= r \left[R^{ij} + dx^i \wedge k^j + k^i \wedge dx^j - r k^i \wedge k^j\right] + (\text{“ ”}) \wedge dr \,.
\end{aligned} \tag{4.11}$$

(The components along $dx^i \wedge dr$ in the second line will not be needed.)

[4] Strictly speaking, $F^{\mu\nu}$ is the SO(5) projection of the SO$(1,5)$ curvature (cf. (2.5)).

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So far we have not made any approximations. We can now use (3.5) from where we derive[5]

$$\begin{aligned} k_{ij} &= \overset{(1)}{g}_{ij} + 2r\,\overset{(2)}{g}_{ij} + \cdots\,, \\ g^{ij} &= \overset{(0)}{g}{}^{ij} - r\,\overset{(1)}{g}{}^{ij} + r^2(-\,\overset{(2)}{g}{}^{ij} + (\overset{(1)}{g}{}^2)^{ij}) + \cdots\,, \\ k^i{}_j &= \overset{(1)}{g}{}^i{}_j + r\left(2\,\overset{(2)}{g}{}^i{}_j - (\overset{(1)}{g}{}^2)^i{}_j\right) + \cdots\,. \end{aligned} \tag{4.12}$$

On the right hand side of these expressions, indices are lowered and raised with $g^{(0)}$.

4.2 The Chern-Simons holographic stress tensor

Consider the Chern-Simons equations of motion (2.10). Our aim is to rederive the holographic stress tensor found in section 3 using these equations. We shall first give an argument based only on the structure of these equations. In the next paragraph we prove, using a hamiltonian approach, that our formula is in fact the variation of the renormalized action with respect to the boundary metric.

We start for illustrative purposes with the five-dimensional case, but we shall see that for Chern-Simons theories of gravity the holographic stress tensor calculation is the same in all (odd) dimensions. We shall in fact provide a formula for this tensor valid on any (integer) dimension d.

Five dimensional CS gravity. The equations of motion were given in (4.1). Let us study them in the lowest non-trivial order, i.e., the equations involving $g^{(0)}$ and $g^{(1)}$. These are

$$\begin{aligned} \varepsilon_{ijkl}F^{ij}\wedge F^{kl} = 0 \quad &\Rightarrow \quad r^2\varepsilon_{ijkl}\left(\overset{(0)}{R}{}^{ij} + 2dx^i\wedge\,\overset{(1)}{g}{}^j\right)\wedge\left(\overset{(0)}{R}{}^{kl} + 2dx^k\wedge\,\overset{(1)}{g}{}^l\right) = 0\,, \\ 4\,\varepsilon_{ijkl}F^{ij}\wedge F^{kr} = 0 \quad &\Rightarrow \quad 4\,r^3\varepsilon_{ijkl}\left(\overset{(0)}{R}{}^{ij} + 2\,dx^i\wedge\,\overset{(1)}{g}{}^j\right)\wedge\,\overset{(0)}{D}\,\overset{(1)}{g}{}^k = 0 \end{aligned} \tag{4.13}$$

where, in the second column, we have rewritten the equations using (4.11) and have kept only the lowest order terms.

Thanks to the Bianchi identity $D^{(0)}\wedge R^{(0)ij} = 0$ the covariant derivative in the second equation can be pulled out to obtain

$$\overset{(0)}{D}\left(4\varepsilon_{ijkl}\left(\overset{(0)}{R}{}^{ij} + dx^i\wedge\,\overset{(1)}{g}{}^j\right)\wedge\,\overset{(1)}{g}{}^k\right) = 0\,. \tag{4.14}$$

We write this equation in the form

$$4\varepsilon_{ijkl}\left(\overset{(0)}{R}{}^{ij} + dx^i\wedge\,\overset{(1)}{g}{}^j\right)\wedge\,\overset{(1)}{g}{}^k = \tilde{T}_l \tag{4.15}$$

[5]In EH gravity for D odd the FG expansion (3.5) needs to be modified. Starting at $\mathcal{O}(\rho^n)\log(\rho)$ terms appear. Without the logarithmic terms the equations of motion for the $g^{(n)}$ are inconsistent. Working in non-integer dimensions, as we did in section 3, does not require the log terms for any gravitational theory. For CS gravity in integer dimensions, at least to the order we are considering here, they do not seem to be necessary either.

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where $\tilde{T}_l$ is an "integration constant" 3-form that must be conserved $D\wedge\tilde{T}_l = 0$. Eq. (4.15) is an algebraic equation for $g^{(1)\,i}$ which, in principle, can be solved, and the solution involves the conserved tensor T_{ij}. The index structure of $\tilde{T}$ is $\tilde{T}^i{}_{npq}$ being antisymmetric in npq. We dualize and define a rank two tensor

$$T^{ij} = \frac{1}{3!}\varepsilon^{inpq}\tilde{T}^j{}_{npq}\,, \tag{4.16}$$

which is symmetric thanks to $g^{(1)}{}_{ij} = g^{(1)}{}_{ji}$ and $R^{(0)}{}_{ijkl} = R^{(0)}{}_{klij}$. Clearly, the conservation equation $D\wedge\tilde{T}^i = 0$ in terms of T^{ij} reads $T^{ij}{}_{;j} = 0$.

The conserved tensor is not completely arbitrary. In fact, the remaining equation of motion (4.13) fixes its trace $g_{ij}T^{ij}$ to be equal to the four-dimensional Euler density. To see this we first note that the trace $g_{ij}T^{ij}$ can also be expressed in terms of the 3-form $\tilde{T}_l$ in a convenient way. Since $^*(dx^i\wedge dx^j\wedge dx^k\wedge dx^l) = \epsilon^{ijkl}$ it follows

$$g_{ij}T^{ij} = {}^*\!\left(dx^i\wedge\tilde{T}_i\right). \tag{4.17}$$

Hence, from the definition of $\tilde{T}_i$ (c.f. (4.15)) and (4.13) we find

$$dx^l\wedge\tilde{T}_l = -4\varepsilon_{ijkl}\left(\overset{(0)}{R}{}^{ij} + dx^i\wedge\overset{(1)}{g}{}^j\right)\wedge\overset{(1)}{g}{}^k\wedge dx^l = \varepsilon_{ijkl}\,\overset{(0)}{R}{}^{ij}\wedge\overset{(0)}{R}{}^{kl} \tag{4.18}$$

or, what is equivalent thanks to (4.17), $g_{ij}T^{ij} = E_4$.

Of course, what we found here is just the set of equations $(3.8)_{2,3}$ and (3.9) evaluated at $d = 4$. In fact, this structure is present for all Chern-Simons theories: the ambiguity in the FG-expansion always occurs in $g^{(1)}$ and it equals the energy-momentum tensor T_{ij}.

A general formula. The analysis of the Chern-Simons equations in other integer dimensions reveals that the same structure appears in all cases. In $2n+1$ dimensions one finds

$$\tilde{T}_i = \int_0^1 dt\, 2n\epsilon_{i_1i_2\ldots i_{2n-1}i}\left[\overset{(0)}{R}{}^{i_1i_2} + 2t dx^{i_1}\wedge\overset{(1)}{g}{}^{i_2}\right]^{n-1}\wedge\overset{(1)}{g}{}^{i_{2n-1}}, \tag{4.19}$$

as can be checked for lower n cases, and we have, in particular, checked $n\le 3$. In (4.19), t is an auxiliary parameter and the symbol $[\;]^{n-1}$ means $n-1$ factors of the tensor $R^{(0)}{}^{ij} + 2t\; dx^i\wedge g^{(1)\,j}$ contracting $2n-2$ indices in the Levi-Cevita symbol. We now check in general that (4.19) is conserved and its trace equal to E_{2n}, as a consequence of the $2n+1$ Chern-Simons equations of motion.

Taking the covariant derivative of $\tilde{T}_i$ the integral over t drops out and we find

$$D\tilde{T}_i = 2n\,\epsilon_{i_1i_2\ldots i_{2n-1}i}\left[\overset{(0)}{R}{}^{i_1i_2} + 2dx^{i_1}\wedge\overset{(1)}{g}{}^{i_2}\right]^{n-2}\wedge D\,\overset{(1)}{g}{}^{i_{2n-1}} \tag{4.20}$$

which is zero thanks to the $2n+1$ Chern-Simons equations. In the same way, using the identity $(a+b)^n = a^n + n\int dt(a+tb)^{n-1}b$ and the equation of motion $\epsilon_{i_1\ldots i_{2n}}[R^{(0)i_1i_2} + 2dx^{i_1}\wedge g^{(1)\,i_2}]^n = 0$, we can compute the trace $dx^i\wedge\tilde{T}_i$ and find the 2n-dimensional Euler density,

$$dx^i\wedge\tilde{T}_i = \epsilon_{i_1\ldots i_{2n}}\,\overset{(0)}{R}{}^{i_1i_2}\wedge\cdots\wedge\overset{(0)}{R}{}^{i_{2n-1}i_{2n}}\,, \tag{4.21}$$

as expected.

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5. Hamiltonian method

We have found in the previous sections a general formula for the stress-tensor for Chern-Simons gravity, via dimensional regularization methods, and by a direct use of the equations of motion in integer dimensions. In this section we would like to rederive this formula as the functional variation of the effective action with respect to the boundary metric,

$$T_{ij} = \frac{2}{\sqrt{g_{(0)}}} \frac{\delta I[g_{(0)}]}{\delta g^{ij}_{(0)}} \,. \tag{5.1}$$

In the AdS/CFT correspondence, the functional I is the regularized and renormalized bulk gravitational action written as a function of $g_{(0)}$.

We shall be interested directly in the stress tensor and not in the effective action. The hamiltonian formalism of gravity provides a shorter method to compute T_{ij} which will be particularly convenient in Chern-Simons gravity.

The computation of holographic anomalies via hamiltonian methods has also been considered in [18] and [19]. We shall briefly discuss the general idea and then apply it to Chern-Simons gravity.

5.1 The method

If the metric is put in the ADM form

$$ds^2 = N^2 dr^2 + h_{ij}(dx^i + N^i dr)(dx^j + N^j dr) \tag{5.2}$$

the variation of the ADM action, evaluated on any solution of the equations of motion is

$$\delta I = \int_{r=0} d^{2n}x \, \pi^{ij} \delta h_{ij} \,. \tag{5.3}$$

In Einstein gravity,

$$\pi^{ij} = \sqrt{h}(K^{ij} - K h^{ij}) \,, \qquad K_{ij} = -\frac{1}{2N} h'_{ij} \,. \tag{5.4}$$

In Chern-Simons gravity (5.3) will still be true, although the relation between the momenta and extrinsic curvature will change. The formula (5.3) gives the variation of the action directly in terms of a boundary integral evaluated at $r = 0$. However, there are two problems with this expression. First, it diverges and needs to be regularized and renormalized. Second, in FG what is fixed is $g^{(0)}{}_{ij}$, not h_{ij}.

The first issue can easily be solved by adding covariant counterterms. We first regularize by evaluating at some fixed finite r. The subtraction will be quite straightforward. The second problem is more delicate, but has a nice solution. We would like the variation of the action to have the form $\int T^{ij}_{(\mathrm{reg})} \delta g_{(0)ij}$. However, replacing in (5.3)

$$h_{ij} = \frac{1}{r} \left(\overset{(0)}{g}_{ij} + r \overset{(1)}{g}_{ij} + r^2 \overset{(2)}{g}_{ij} + \cdots \right) \tag{5.5}$$

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we get

$$\delta I = \int_r d^{2n}x \left(\frac{1}{r} \pi^{ij} \delta \overset{(0)}{g}_{ij} + \pi^{ij} \delta \overset{(1)}{g}_{ij} + \cdots \right). \tag{5.6}$$

Now, the point is that the extra terms, $\int \pi^{ij}\, \delta g^{(1)}{}_{ij} \ldots$, can be transformed into contributions of the form $\int A^{ij}\, \delta g^{(0)}{}_{ij}$ by making appropriated "integral by parts", and discarding total variations.

Our prescription is then to expand (5.6) and make the necessary "integral by parts" until it has the form $\int T^{ij}_{(\mathrm{reg})}\, \delta g^{(0)}{}_{ij}$, plus total variations. Then, we discard all total variations, identify $T^{ij}_{(\mathrm{reg})}$, renormalize by subtracting the divergent terms, and find $T^{ij}_{(\mathrm{ren})}$.

The terms which are total variations, $\delta f(g^{(0)}, g^{(1)}, g^{(2)}, \ldots)$, must be discarded because they cannot be written, by means of integrals by parts, as $A^{ij}\, \delta g^{(0)}$. Hence, the Dirichlet problem dictates that we add to the action a boundary term $-f$ to cancel this variation.

As a warm-up we will first treat the standard Einstein action. We should and will recover the energy momentum tensor found in [20].[6] From (5.4) and (4.9) we get

$$\delta I = \int \sqrt{g} \left[-\frac{1}{r^{d/2-1}} (k^{ij} - k g^{ij}) + \frac{1-d}{r^{d/2}} g^{ij} \right] \delta g_{ij} = -\int \frac{\sqrt{g}}{r^{d/2-1}} (k^{ij} - k g^{ij}) \delta g_{ij} \tag{5.7}$$

where the second term has been discarded because $\sqrt{g} g^{ij} \delta g_{ij} = 2\delta \sqrt{g}$ is a total variation.

Consider first $d+1 = 3$. Since $r^{d/2-1} = 1$ in this case, the variation of the action δI is finite. And since $k_{ij} = g^{(1)}{}_{ij} + \cdots$, its non-zero part is,

$$\delta I = -\int \sqrt{\overset{(0)}{g}} \Big(\overset{(1)}{g}{}^{ij} - \mathrm{Tr}(\overset{(1)}{g})\, \overset{(0)}{g}{}^{ij} \Big) \delta \overset{(0)}{g}_{ij} \tag{5.8}$$

giving the correct expression for T^{ij} (see eq. (3.10) in [20]).

Consider now $d+1 = 5$. In this case there is a divergent piece that we cancel by a subtraction. We focus on the finite piece obtained by expanding $\sqrt{g}(k^{ij} - k g^{ij}) \delta g_{ij}$ to $\mathcal{O}(r)$. It is useful to note that $\sqrt{g} k g^{ij} \delta g_{ij} = 2k \delta \sqrt{g}$. The finite piece in the variation of the action is then

$$\delta I_{\mathrm{finite}} = \int \left[\overset{(1)}{\sqrt{g}}\, \overset{(0)}{k}{}^{ij}\, \delta \overset{(0)}{g}_{ij} + \overset{(0)}{\sqrt{g}}\, \overset{(1)}{k}{}^{ij}\, \delta \overset{(0)}{g}_{ij} + \overset{(0)}{\sqrt{g}}\, \overset{(0)}{k}{}^{ij}\, \delta \overset{(1)}{g}_{ij} - 2\Big(\overset{(0)}{k}\, \delta \overset{(1)}{\sqrt{g}} + \overset{(1)}{k}\, \delta \overset{(0)}{\sqrt{g}} \Big) \right]. \tag{5.9}$$

This explicitly involves variations of $g^{(1)}{}_{ij}$. These variations can be transformed into variations of $g^{(0)}{}_{ij}$ by performing "integrals by parts". We give the details of one term. Recalling that $k^{(0)}{}_{ij} = g^{(1)}{}_{ij}$ the third term is

$$\begin{aligned}
\overset{(0)}{\sqrt{g}}\, \overset{(1)}{g}{}^{ij} \delta\, \overset{(1)}{g}_{ij} &= \sqrt{\overset{(0)}{g}}\, \overset{(0)}{g}{}^{ik}\, \overset{(0)}{g}{}^{jl}\, \overset{(1)}{g}_{kl}\, \delta \overset{(1)}{g}_{ij} \\
&= \frac{1}{2} \sqrt{\overset{(0)}{g}}\, \overset{(0)}{g}{}^{ik}\, \overset{(0)}{g}{}^{jl}\, \delta \Big(\overset{(1)}{g}_{kl}\, \overset{(1)}{g}_{ij} \Big) \\
&= -\frac{1}{2} \delta \Big(\sqrt{\overset{(0)}{g}}\, \overset{(0)}{g}{}^{ik}\, \overset{(0)}{g}{}^{jl} \Big)\, \overset{(1)}{g}_{kl}\, \overset{(1)}{g}_{ij} + \text{total variation}
\end{aligned}$$

[6] We do not include the log terms. Including then would simply mean a finite renormalization and it does not affect the trace of T_{ij}.

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$$= -\frac{1}{2}\delta\Big(\sqrt{\overset{(0)}{g}}\Big)\,\mathrm{Tr}\Big(\overset{(1)}{g}{}^2\Big) - \sqrt{\overset{(0)}{g}}\;\delta\,\overset{(0)}{g}{}^{ij}\Big(\overset{(1)}{g}{}^2\Big)_{ij}$$

$$= \sqrt{\overset{(0)}{g}}\left[\frac{1}{4}\,\mathrm{Tr}\Big(\overset{(1)}{g}{}^2\Big)\;\overset{(0)}{g}_{ij} - \Big(\overset{(1)}{g}{}^2\Big)_{ij}\right]\delta\,\overset{(0)}{g}{}^{ij}\,. \tag{5.10}$$

Proceeding in this way, all variations of $g^{(1)}{}_{ij}$ can be transformed into variations of $g^{(0)}{}_{ij}$. Up to total variations we finally get

$$\delta I = \int \sqrt{\overset{(0)}{g}}\left[\overset{(2)}{g}_{ij} - \frac{1}{8}\,\overset{(0)}{g}_{ij}\left(\Big(\mathrm{Tr}\;\overset{(1)}{g}\Big)^2 - \mathrm{Tr}\Big(\overset{(1)}{g}{}^2\Big)\right) - \frac{1}{2}\Big(\overset{(1)}{g}{}^2\Big)_{ij} + \frac{1}{4}\;\overset{(1)}{g}_{ij}\;\mathrm{Tr}\;\overset{(1)}{g}\right]\delta\,\overset{(0)}{g}{}^{ij}\,. \tag{5.11}$$

in full agreement with [20]. (Here we have used one equation of motion $\mathrm{Tr}(g^{(2)}) = \frac{1}{4}\,\mathrm{Tr}\;g^{(1)2}$ only to make contact with [20]. The above prescription certainly does not require to use the solution to the equations of motion.)

5.2 Chern-Simons gravity in hamiltonian form and its stress-tensor

We now apply the above method to Chern-Simons gravity. The hamiltonian form of "Lovelock" theories of gravity was worked out in [21]. As we mentioned before, Chern-Simons gravity is a particular family of theories on which all coefficients are correlated. This has the effect of enlarging the local symmetry group from SO(5) to SO(1, 5) (in five euclidean dimensions).

To apply our method of stress-tensor renormalization, we write the variation of the action as

$$\delta I = \int \pi^{ij}\,\delta h_{ij}$$

where the relation between the momenta π^{ij} and the extrinsic curvature for the general Lovelock action is [21],

$$\pi^i{}_j = -\frac{1}{4}\sqrt{g}\sum_{p\geq 0}\alpha_p\sum_{s=0}^{p-1} C_{s(p)}\,\delta^{[i_1\ldots i_{2s}\ldots i_{2p-1}i]}_{[j_1\ldots j_{2s}\ldots j_{2p-1}j]}\hat{R}^{j_1j_2}{}_{i_1i_2}\ldots\hat{R}^{j_{2s-1}j_{2s}}{}_{i_{2s-1}i_{2s}}K^{j_{2s+1}}{}_{i_{2s+1}}\ldots K^{j_{2p-1}}{}_{i_{2p-1}} \tag{5.12}$$

where

$$C_{s(p)} = \frac{(-4)^{p-s}}{s![2(p-s)-1]!!} \tag{5.13}$$

and the hatted tensors refer to $d+1$ dimensional ones (c.f. (4.5)).

The coefficients α_p depend on the theory under consideration. For Chern-Simons gravity they have to be chosen as

$$\alpha_p = \frac{n![2(n-p)]!}{2^{p-1}(n-p)!}\,. \tag{5.14}$$

In five dimensions the sum in (5.12) contains three terms with coefficients

$$\alpha_0 = 2\times 4!\,,\qquad \alpha_1 = 4\,,\qquad \alpha_2 = 1\,. \tag{5.15}$$

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Inserting them in (5.12) we get[7]

$$\pi^i{}_j = \sqrt{h}\left[4\delta^{[ni]}_{[qj]}K^q_n + \delta^{[nmpi]}_{[qksj]}\left(-\frac{2}{3}K^q_n K^k_m K^s_p + R^{qk}{}_{nm}K^s_p\right)\right]. \tag{5.16}$$

Next we write this expression in terms of the FG metric g_{ij} defined as $h_{ij} = \frac{1}{r}g_{ij}$. Using eq. (4.9) we find

$$\begin{aligned}\pi^i{}_j \; = \; & \frac{\sqrt{g}}{r^2}\Bigg[4\delta^{[ni]}_{[qj]}(\delta^q_n - rk^q_n) + \delta^{[nmpi]}_{[qksj]}\Bigg(-\frac{2}{3}(\delta^q_n - rk^q_n)(\delta^k_m - rk^k_m)(\delta^s_p - rk^s_p) + \\ & + rR^{qk}{}_{nm}(\delta^s_p - rk^s_p)\Bigg)\Bigg]. \end{aligned} \tag{5.17}$$

This expression is much more manageable that it appears. We need to look at $\pi^{ij}\delta h_{ij} = \pi^i{}_n h^{ni}\delta h_{ij} = \pi^i{}_n g^{ni}\delta g_{ij}$. *Without making any approximations yet*, we look at the different powers of r in this expression and conclude:

- The coefficient of $1/r^2$ (the piece containing only Kronecker deltas) gives $\delta(\sqrt{g})$ and hence we discard it.

- The coefficient of $1/r$ has two contributions. A piece linear in k multiplied by zero! In fact, there is a cancellation between the linear and the cubic terms which, of course, happens only for the Chern-Simons action whose coefficients are correlated. There is also a piece linear in the curvature. However, it is direct to see that this piece is proportional to the Einstein tensor of the metric g_{ij}, hence it can be written as $\delta(\sqrt{g}R)$, and we discard it as well.

- Finally, the coefficient of $r^0 = 1$ is

$$\delta I = -2\int \sqrt{g}\delta^{[nmpi]}_{[qksj]}\left(\delta^q_n k^k{}_m k^s{}_p + \frac{1}{2}R^{qk}{}_{nm}k^s{}_p\right)g^{jn}\delta g_{in}\,. \tag{5.18}$$

Since this term occurs at order zero, its non-zero contribution is obtained simply by replacing $g \to g_{(0)}$ and $k \to g_{(1)}$. The coefficient is by definition the stress-tensor and it gives exactly the component version of (4.15).

5.3 The black hole mass

As a final application of our formula for the stress tensor let us check that it provides the correct value of the mass for the Chern-Simons black holes. Black holes for Chern-Simons gravity exists and have been found in [22]. The metric in five dimensions is

$$ds^2 = -(r^2 - c)dt^2 + \frac{dr^2}{r^2 - c} + r^2 d\Omega_3 \tag{5.19}$$

where c is an integration constant related to the mass M in a nonlinear way,

$$c = -1 + \sqrt{1 + \kappa M}\,, \tag{5.20}$$

[7] In [21] the signature $-,+,+,\ldots$ was assumed. A quick way to transfer the time coordinate into h_{ij} is to set $N \to iN$ (hence $K \to -iK$) and $\sqrt{h} \to i\sqrt{h}$.

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and κ is a constant that depends on the normalization for Newton's constant. This expression for M was obtained in [22] using the standard ADM procedure. The minus sign in front of the square root provides a solution as well but it is not a black hole. See [23] for other implications of the "wrong sign".

We now put this metric in the FG form. This is achieved by the simple radial redefinition,

$$r \to \rho : \quad r = \frac{1+\rho c}{2\sqrt{\rho}} \tag{5.21}$$

which brings the metric into the FG form

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\left[\frac{1}{4}(-dt^2 + d\Omega_3) + \frac{c\rho}{2}(dt^2 + d\Omega_3) + \frac{c^2\rho^2}{4}(-dt^2 + d\Omega_3)\right]. \tag{5.22}$$

Note that only three terms in the FG expansion are non zero. The boundary metric $g^{(0)}$ ($\Re \times S_3$) has a vanishing Weyl tensor, and hence this is consistent with [24]. The mass, defined as the integral of T_0 at the sphere at infinity is given by

$$M = \int_{S_3} \tilde{T}_0 = \int_{S_3} 4\epsilon_{\alpha\beta\gamma}\left(\overset{(0)}{R}{}^{\alpha\beta} + dx^\alpha \wedge \overset{(1)}{g}{}^{\beta}\right) \wedge \overset{(1)}{g}{}^{\gamma} \tag{5.23}$$

where we have denoted the coordinates on the sphere by x^α. (Here it is convenient to work directly with the 3-form $\tilde{T}_i$). The curvature on S_3 is $R^{\alpha\beta} = 4dx^\alpha \wedge dx^\beta$ and, from (5.22) we find ${g^{(1)}}_{\alpha\beta} = 2c\, {g^{(0)}}_{\alpha\beta}$, which implies $g^{(1)\,\alpha} = 2cdx^\alpha$. Replacing in the formula for M we find (Ω_3 is the volume of the three-sphere),

$$M = 4\Omega_3(4+2c)(2c) = 16\Omega_3(2c + c^2) \tag{5.24}$$

which is equivalent to the non-linear relation (5.20).

Using the general formula (4.19) we could also find the mass for a generic theory in any number of dimensions. We do not reproduce the result here which has been found using the standard ADM formalism in [22] (see [25] for a recent discussion).

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